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ANALYTICAL SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR THE ELLIPSOIDAL STATISTICAL EQUATION

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This paper describes an analytical method for solving semispatial boundary-value problems for the ellipsoidal statistical equation with a frequency proportional to the molecular velocity. The classical Smoluchowski problem of a temperature jump in a rarefied gas and weak vaporization (condensation) is solved. Numerical calculations of the obtained expressions are performed. A comparison is made with previous results.

Key words: *statistical equation, Smoluchowski problem, Riemann–Hilbert boundary-value problem.*

Introduction. Model kinetic equations are still widely used in the kinetic theory of gases (see, for example, [1–3]).

The well-known kinetic Bhatnagar–Gross–Krook (BGK) equation leads to an incorrect Prandtl number. This drawback is eliminated by using higher-order models (in analytical solutions), such as the Shakhov equation and the ellipsoidal statistical equation (ES equation) or the full Boltzmann equation in numerical solutions.

For all model equations with a constant collision frequency $\nu = \text{const}$, analytical methods [4–7] of solving boundary-value problems have been developed.

Alongside kinetic equations with $\nu = \text{const}$, equations with a collision frequency proportional to the molecular velocity are used. Such equations correspond to the more adequate hypothesis on the constancy of the free path length of molecules $l = \text{const}$. In [8], we showed that in slip problems, the ES equation for $l = \text{const}$ gives results close to those obtained using the full Boltzmann equation for rigid sphere molecules. In the same study, we developed an analytical solution method for the ES equation as applied to slip problems. So far, there is no method for the analytical solution of general boundary-value problems for the ES equation. Among such problems is the Smoluchowski problem, which combines the problems of a temperature jump and weak vaporization (condensation). To fill this gap, in the present paper, we develop an analytical method for solving semispatial boundary-value problems for the ES equation with a collision frequency $\nu = \nu_0 V$, where $V = \sqrt{V_1^2 + V_2^2 + V_3^2}$ is the molecular velocity magnitude. The exact solution of the Smoluchowski problem was obtained.

The problem of a temperature jump in a gas is among the most important problems of the interaction of a gas with a solid (or a condensed phase). This problem has been a subject of extensive research using both numerical and analytical methods. Because of the fundamental nature of the problem considered, the interest in analytical methods remains high (see [3] and references therein).

The analytical results available on this problem were obtained using the BGK equation (with constant and variable collision frequencies) and the ES equation with $\nu = \text{const}$. It seems urgent to develop an analytical method for the ES equation for the case $l = \text{const}$ and to employ it to solve the Smoluchowski problem. It is important to bear in mind that analytical methods give a complete solution of the problem since they allow one to obtain not only magnitudes of jumps of macroparameters (temperature and concentration) but also the complete distribution function.

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1. Formulation of the Problem and the Basic Equations. Let us consider the stationary linearized ES equation with frequency $\nu = \nu_0 V$ (see [4, 9]) in dimensionless variables:

$$\boldsymbol{C}\nabla\varphi + C\varphi(\boldsymbol{r}, \boldsymbol{C}) = \frac{\sqrt{\pi}}{2}\, C \int \rho(C')k(\boldsymbol{C}, \boldsymbol{C}')\varphi(\boldsymbol{r}, \boldsymbol{C}')\, d^3C'. \tag{1.1}$$

Here $\boldsymbol{C} = \sqrt{m/(2kT_s)}\boldsymbol{V}$ (T_s is the surface temperature and k is the Boltzmann constant) is the dimensionless molecular velocity and $\boldsymbol{r}' = \nu_0 \boldsymbol{r}$ is the dimensionless coordinate (here and below, the prime at the dimensionless coordinate is omitted).

The kernel of Eq. (1.1) is defined by the expression

$$k(\boldsymbol{C}, \boldsymbol{C}') = 1 + \frac{3}{2}\,\boldsymbol{C}\boldsymbol{C}' + \frac{1}{2}\,(C^2 - 2)(C'^2 - 2) + \gamma \sum_{i,j=1}^{3} \Big(C_iC_j - \frac{1}{3}\,\delta_{ij}C^2\Big)\Big(C_i'C_j' - \frac{1}{3}\,\delta_{ij}C'^2\Big),$$

where $\rho(C) = \pi^{-3/2} C \exp(-C^2)$; γ is a parameter that can be found from the definition of the Prandtl number; δ_{ij} is the Kronecker delta; $\delta_{ii} = 1$; $\delta_{ij} = 0$, and $i \neq j$.

We note that for $\gamma = 0$, Eq. (1.1) becomes the BGK equation because the ES kernel k becomes the BGK kernel

$$k_0(\boldsymbol{C}, \boldsymbol{C}') = 1 + \frac{3}{2}\,\boldsymbol{C}\boldsymbol{C}' + \frac{1}{2}\,(C^2 - 2)(C'^2 - 2).$$

We consider the class of problems in which the distribution function depends on the space variable x and shows isotropy in the plane $C_1 = \text{const}$, C_2, C_3. For these problems, all nondiagonal components of the tensor ($i \neq j$)

$$\int \rho(C)\Big(C_iC_j - \frac{1}{3}\delta_{ij}C^2\Big)\varphi(x, \boldsymbol{C})\, d^3C$$

are equal to zero. In addition, under these assumptions, the function $\varphi(x, \boldsymbol{C})$ depends only on x, C, and $\mu = C_1/C$. Therefore, Eq. (1.1) is simplified:

$$\mu\,\frac{\partial\varphi}{\partial x} + \varphi(x, \mu, C) = \int_{-1}^{1} d\mu' \int_{0}^{\infty} \exp(-C'^2)C'^3 k(\mu, C; \mu', C')\varphi(x, \mu', C')\, dC', \tag{1.2}$$

where

$$k(\boldsymbol{C}, \boldsymbol{C}') = k_0(\boldsymbol{C}, \boldsymbol{C}') + \frac{3}{2}\,\gamma C^2 C'^2 \Big(\mu^2 - \frac{1}{3}\Big)\Big(\mu'^2 - \frac{1}{3}\Big).$$

We use the definition of the Prandtl number: $\Pr = 5k\eta/(2m\text{æ})$. Here m is the molecular weight, η is the viscosity coefficient, and æ is the thermal conductivity. Expressing the viscosity and heat conductivity in terms of the parameter γ, we obtain

$$\gamma = \frac{40(9\Pr - 8)}{288\Pr - 256 + 75\pi}.$$

For the frequently used value of the Prandtl number $\Pr = 2/3$, this formula yields $\gamma = -0.466148$.

In the Smoluchowski problem, a gas occupies a half-space $x > 0$ above a flat wall from which vaporization (condensation) of the gas (vapor) molecules occurs and there is heat transfer between the condensed phase and the gas (vapor). We assume that away from the surface there are a temperature gradient perpendicular to the surface (and the corresponding heat flux) and a certain mass average velocity of the gas directed from or to the surface (vaporization or condensation), i.e., $T(x) = T_0 + K_t x$, $n(x) = n_0 - (n_0K_t/T_s)x$, $K_t = (dT/dx)_\infty$, $\boldsymbol{U}(x) = \{U_\infty, 0, 0\}$, and $x \to +\infty$. The Smoluchowski problem consists of finding the relative temperature jump $\varepsilon_t = (T_0 - T_s)/T_s$ as a function of the relative temperature gradient $k_t = K_t/T_s$ and the rate of vaporization (condensation) $U = \sqrt{m/(2kT_s)}\,U_\infty$. Taking into account the linear nature of the problem, we can write $\varepsilon_t = T_tk_t + T_uU$. The nondimensional quantities T_t and T_u are called the temperature jump coefficients. Another important characteristic of the gas is the relative jump in the concentration $\varepsilon_n = (n_0 - n_s)/n_s$ (n_s is the saturated-vapor concentration at a temperature T_s) for which $\varepsilon_n = N_tk_t + N_uU$ (N_t and N_u are the concentration jump coefficients).

Assuming that the reflection of the molecules from the wall is purely diffusive, we formulate the boundary conditions in the Smoluchowski problem:

$$\begin{aligned} &\varphi(0,\mu,C)=0, \qquad 0<\mu<1, \\ &\varphi(x,\mu,C)=\varphi_{as}(x,\mu,C)+o(1), \qquad x\to+\infty, \quad -1<\mu<0, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\varphi(0,\mu,C)=0, \qquad 0<\mu<1, \\ &\varphi(x,\mu,C)=\varphi_{as}(x,\mu,C)+o(1), \qquad x\to+\infty, \quad -1<\mu<0, \end{aligned} \tag{1.3}$$

where

$$\varphi_{as}=\varepsilon_n+2U\mu C+\Bigl(C^2-\frac{3}{2}\Bigr)\varepsilon_t+k_t\Bigl[(x-\mu)\Bigl(C^2-\frac{5}{2}\Bigr)-\frac{2}{3\sqrt{\pi}}\,\mu C\Bigr].$$

Equation (1.2) has four particular solutions: three of them are collision invariants 1, μC, and C^2, and the fourth solution $(x-\mu)(C^2-5/2)-2\mu C/(3\sqrt{\pi})$ describes heat transfer in the nonuniformly heated gas.

Taking into account the structure of the kernel of Eq. (1.2), we seek a solution of the problem (1.2), (1.3) in the form

$$\varphi(x,\mu,C)=h_1(x,\mu)+Ch_2(x,\mu)+(C^2-2)h_3(x,\mu).$$

The resulting problem consists of the equation

$$\mu\,\frac{\partial h}{\partial x}+h(x,\mu)=\frac{1}{2}\int\limits_{-1}^{1}K(\mu,\mu')h(x,\mu)\,d\mu' \tag{1.4}$$

and the boundary conditions

$$\begin{aligned} &h(0,\mu)=0, \qquad 0<\mu<1, \\ &h(x,\mu)=h_{as}(x,\mu)+o(1), \qquad x\to+\infty, \quad -1<\mu<0. \end{aligned} \tag{1.5}$$

Here $h=\text{col}\,\{h_1(x,\mu),h_2(x,\mu),h_3(x,\mu)\}$ is a column vector and

$$h_{as}(x,\mu)=\text{col}\,\Bigl\{\varepsilon_n+\frac{1}{2}\,\varepsilon_t-\frac{1}{2}\,k_t(x-\mu),\Bigl(2U-\frac{2}{3\sqrt{\pi}}\,k_t\Bigr)\mu,\varepsilon_t+k_t(x-\mu)\Bigr\};$$

the kernel of the equation is

$$K(\mu,\mu')=K_0+3\mu\mu' K_1+3\gamma\Bigl(\mu^2-\frac{1}{3}\Bigr)\Bigl(\mu'^2-\frac{1}{3}\Bigr)K_2,$$

where

$$K_0=\begin{bmatrix}1&4\alpha&0\\0&0&0\\0&\alpha&1\end{bmatrix},\qquad K_1=\begin{bmatrix}0&0&0\\2\alpha&1&\alpha\\0&0&0\end{bmatrix},\qquad K_2=\begin{bmatrix}2&10\alpha&2\\0&0&0\\1&5\alpha&1\end{bmatrix}\qquad \Bigl(\alpha=\frac{3}{16}\sqrt{\pi}\Bigr).$$

2. Separation of Variables, Eigenvectors, and Eigenvalues. Separation of variables in Eq. (1.4) using the general Fourier method leads to the solutions $h_\eta(x,\mu)=\exp(-x/\eta)\Phi(\eta,\mu)$, in which η is a spectral parameter, or the separation parameter, and the vector Φ is a solution of the characteristic equation

$$(\eta-\mu)\Phi(\eta,\mu)=\frac{1}{2}\,\eta D(\mu,\eta)n(\eta),\qquad n(\eta)=\text{col}\,\{n_1(\eta),n_2(\eta),n_3(\eta)\}=\int\limits_{-1}^{1}\Phi(\eta,\mu)\,d\mu.$$

Here

$$D(\mu,\eta)=D_0(\mu\eta)-\gamma(\mu^2-1/3)D_1(\eta),\qquad d(\eta)=1+3c\eta^2;$$

$$D_0=\begin{bmatrix}1&4\alpha&0\\0&3c\mu\eta&0\\0&\alpha&1\end{bmatrix};\qquad D_1(\eta)=\begin{bmatrix}2&10\alpha d(\eta)&2\\0&0&0\\1&5\alpha d(\eta)&1\end{bmatrix};\qquad c=1-9\alpha^2.$$

For $\eta \in (-1, 1)$, the solution of the characteristic equation is taken in the space of generalized functions [10]:

$$\Phi(\eta, \mu) = F(\eta, \mu) n(\eta).$$

Here

$$F(\eta, \mu) = \frac{1}{2} D(\mu, \eta) P \frac{1}{\eta - \mu} + \Lambda(\eta) \delta(\eta - \mu)$$

is the natural matrix function, the symbol Px^{-1} denotes the distribution (the principal value of the integral of x^{-1}), $\delta(x)$ is the delta-function, $\Lambda(z) = \Lambda_0(z) - \gamma \omega_*(z) D_1(z)$ is the dispersion matrix, where

$$\omega_*(z) = 1/3 + (z^2 - 1/3)\lambda_0(z),$$

$$\Lambda_0(z) = \begin{bmatrix} \lambda_0(z) & 4\alpha T(z) & 0 \\ 0 & \omega(z) & 0 \\ 0 & \alpha T(z) & 0 \end{bmatrix},$$

$\lambda_0(z) = 1 + T(z)$ is the dispersion Case function [11], $T(z) = \frac{1}{2} z \int\limits_{-1}^{1} \frac{du}{u - z}$, and $\omega(z) = 1 + 3cz^2 \lambda_0(z)$.

Below, we shall need the representation of the dispersion matrix in the form

$$\Lambda(z) = \lambda_0(z) D(z) - M(z)/3,$$

where

$$M(z) = \begin{bmatrix} 2\gamma & 2\alpha(6 + 5\gamma d(z)) & 2\gamma \\ 0 & -3 & 0 \\ \gamma & \alpha(3 + 5\gamma d(z)) & \gamma \end{bmatrix}.$$

The dispersion function of this problem has the form

$$\lambda(z) = \det \Lambda(z) = \gamma \lambda_0(z) \omega(z) \omega_1(z),$$

where $\omega_1(z) = \gamma^{-1}\lambda_0(z) - 3\omega_*(z) = \lambda_0(z) s(z) - 1$ and $s(z) = -3z^2 + 1 + \gamma^{-1}$.

By the definition (see, for example, [4, 12]) the discrete spectrum of the problem consists of the set of zeroes of the dispersion function. The zero of $\lambda_0(z)$ is (see [11]) the point $z = \infty$ of multiplicity 2, and the zeroes of $\omega(z)$ are (see [4, 9]) the two points $\pm\eta_0$ ($\eta_0 = 1 + 1.12 \cdot 10^{-48}$). From the expansion

$$\gamma \omega_1(z) = \frac{4\gamma - 5}{15 z^2} + \frac{8\gamma - 7}{35 z^4} + \cdots \qquad (z \to \infty)$$

it is obvious that $\omega_1(z)$ has a zero of multiplicity 2 at the point $z = \infty$. The use of the argument principle [13] shows that $\omega_1(z)$ does not have other zeroes. The zero of η_0 corresponds to the eigensolution

$$h_0(x, \mu) = \exp\Big(-\frac{x}{\eta}\Big) \frac{1}{2} \eta_0 D(\mu, \eta_0) \frac{1}{\eta_0 - \mu} n(\eta_0).$$

Substituting this solution into Eq. (1.4), we find that the vector $n(\eta_0)$ is determined from the equation

$$\Lambda(\eta_0) n(\eta_0) = \mathbf{0}. \tag{2.1}$$

By virtue of the equalities

$$\omega(\eta_0) = 1 + 3c\eta_0^2 \lambda_0(\eta_0) = 0, \qquad d(\eta_0)\lambda_0(\eta_0) = T(\eta_0),$$

$$\lambda_0(\eta_0) - 3\gamma\omega_*(\eta_0) = \gamma\omega_1(\eta_0),$$

the matrix $\Lambda(z)$ at the point η_0 can be written as

$$\Lambda(\eta_0) = \gamma \begin{bmatrix} \omega_1(\eta_0) + \omega_*(\eta_0) & 4\alpha d(\eta_0)[\omega_1(\eta_0) + \omega_*(\eta_0)/2] & -2\omega_*(\eta_0) \\ 0 & 0 & 0 \\ -\omega_*(\eta_0) & \alpha d(\eta_0)[\omega_1(\eta_0) - 2\omega_*(\eta_0)] & \omega_1(\eta_0) + 2\omega_*(\eta_0) \end{bmatrix}.$$

Setting $n_2(\eta_0) = \lambda_0(\eta_0)$, from Eq. (2.1), we obtain two equations

$$[\omega_1(\eta_0) + \omega_*(\eta_0)]n_1(\eta_0) - 2\omega_*(\eta_0)n_3(\eta_0) = -4\alpha T(\eta_0)[\omega_1(\eta_0) + \omega_*(\eta_0)/2],$$

$$-\omega_*(\eta_0)n_1(\eta_0) + [\omega_1(\eta_0) + 2\omega_*(\eta_0)]n_3(\eta_0) = -\alpha T(\eta_0)[\omega_1(\eta_0) - 2\omega_*(\eta_0)].$$

From these equations it follows that $n_1(\eta_0) = -4\alpha T(\eta_0)$ and $n_3(\eta_0) = -\alpha T(\eta_0)$. Thus, the vector $n(\eta_0)$ is constructed:

$$n(\eta_0) = \text{col}\,\{-4\alpha T(\eta_0), \lambda_0(\eta_0), -\alpha T(\eta_0)\}.$$

We note that $D_1(\eta_0)n(\eta_0) = \mathbf{0}$, and, hence,

$$D(\mu, \eta_0)n(\eta_0) = [D_0(\mu\eta_0) - \gamma(\mu^2 - 1/3)D_1(\eta_0)]n(\eta_0) = D_0(\mu\eta_0)n(\eta_0) = \text{col}\,\{4\alpha, -\mu/\eta_0, \alpha\}.$$

Thus, the last particular solution is constructed:

$$h_0(x, \mu) = \frac{1}{2}\,\frac{\exp(-x/\eta_0)}{\eta_0 - z}\ \text{col}\,\{4\alpha\eta_0, -\mu, \alpha\eta_0\}.$$

We note that a linear combination of the four particular solutions of Eq. (1.4) corresponding to the point $z = \infty$ constitutes a vector $h_{as}(x, \mu)$.

3. Homogeneous Boundary-Value Problem. Below, we shall need the solution of the Riemann–Hilbert vector homogeneous boundary-value problem

$$X^+(\mu) = G(\mu)X^-(\mu), \qquad G(\mu) = [\Lambda^+(\mu)]^{-1}\Lambda^-(\mu), \qquad 0 < \mu < 1. \tag{3.1}$$

The matrix factor $G(\mu)$ can be written as

$$G(\mu) = [P^+(\mu)]^{-1}P^-(\mu),$$

where $P(z) = \Lambda(z)D^{-1}(z, z)$; $X(z)$ is an unknown matrix and $X^\pm(\mu)$ are boundary values from above/below in the interval $(0, 1)$.

It is clear that $P(z) = \lambda_0(z)E - M(z)D^{-1}(z, z)/3$ (E is a unit matrix) or $P(z) = \lambda_0(z)E - E_1(z)/(3s(z))$, where

$$E_1(z) = \begin{bmatrix} 2 & e_1(z) & 2 \\ 0 & e_2(z) & 0 \\ 1 & e_3(z) & 1 \end{bmatrix}, \qquad \begin{aligned} e_1(z) &= 2\alpha(5 - 2e_2(z)), \\ e_2(z) &= -s(z)/(cz^2), \\ e_3(z) &= \alpha(5 - e_2(z)). \end{aligned}$$

The matrix $P(z)$ is analytic in the complex plane except at the points of the cut $[0, 1]$ and at the simple imaginary poles $\pm i\eta_1$ [$\eta_1 = \sqrt{-1/(3\gamma) - 1/3}$], which are zeroes of $s(z)$. When $\gamma \to 0$ (the ES equation becomes the BGK equation), the poles $\pm i\eta_1$ vanish moving to infinity along the imaginary axis.

To reduce the matrix $P(z)$ to diagonal form, it suffices to reduce the matrix $E_1(z)$ to diagonal form. Considering the eigenvalue problem for the matrix $E_1(z)$, we construct the diagonalizing matrix S:

$$S = \begin{bmatrix} 1 & -4\alpha & 2 \\ 0 & 1 & 0 \\ -1 & -\alpha & 1 \end{bmatrix}, \qquad S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2\alpha & -2 \\ 0 & 3 & 0 \\ 1 & 5\alpha & 1 \end{bmatrix}.$$

From the definition of the matrix S, it follows that $S^{-1}E_1(z)S = \text{diag}\,\{0, e_2(z), 3\}$. We shall seek a solution of problem (3.1) in the form $X(z) = SX_0(z)S^{-1}$, where $X_0(z) = \text{diag}\,\{U(z), V(z), W(z)\}$ is an unknown matrix. Taking into account the diagonalization, we obtain the matrix boundary-value problem

$$\Omega^+(\mu)X_0^+(\mu) = \Omega^-(\mu)X_0^-(\mu), \qquad 0 < \mu < 1, \tag{3.2}$$

where $\Omega(z) = S^{-1}P(z)S = \text{diag}\,\{\lambda_0(z), \omega(z)/(3cz^2), \omega_1(z)/s(z)\}$.

The matrix boundary-value problem (3.2) is now equivalent to the following three scalar boundary-value problems:

$$\begin{aligned} U^+(\mu) &= [\lambda_0^-(\mu)/\lambda_0^+(\mu)]U^-(\mu), \qquad 0 < \mu < 1, \\ V^+(\mu) &= [\omega^-(\mu)/\omega^+(\mu)]V^-(\mu), \qquad 0 < \mu < 1, \\ W^+(\mu) &= [\omega_1^-(\mu)/\omega_1^+(\mu)]W^-(\mu), \qquad 0 < \mu < 1. \end{aligned}$$

The first two problems were already solved in [9], and the third problem is solved similarly to the first. We give the solutions of all problems:

$$U(z) = z \exp(-u(z)), \qquad V(z) = z \exp(-v(z)), \qquad W(z) = z \exp(-w(z)),$$

$$u(z) = \frac{1}{\pi} \int_0^1 \frac{\zeta_0(u)\, du}{u - z}, \qquad v(z) = \frac{1}{\pi} \int_0^1 \frac{\zeta(u)\, du}{u - z}, \qquad w(z) = \frac{1}{\pi} \int_0^1 \frac{\zeta_1(u)\, du}{u - z}.$$

Here

$$\zeta_0(u) = -\frac{\pi}{2} - \arctan\left[\frac{2\lambda_0(u)}{\pi u}\right], \qquad \zeta(u) = -\frac{\pi}{2} - \arctan\left[\frac{2\omega(u)}{3c\pi u^3}\right],$$

$$\zeta_1(u) = -\frac{\pi}{2} - \arctan\left[\frac{2}{\pi u}\Big(\lambda_0(u) - \frac{1}{s(u)}\Big)\right].$$

Thus, the matrix $X(z)$ is constructed and is defined in explicit form by the equality

$$X(z) = \begin{bmatrix} U + 2W & 2\alpha(U - 6V + 5W) & -2U + 2W \\ 0 & 2V & 0 \\ -U + W & \alpha(-2U - 3V + 5W) & 2U + W \end{bmatrix}.$$

We note that $X(z) = zE + X^{(0)} + o(1)$ as $z \to \infty$, where

$$X^{(0)} = -\frac{1}{3} \begin{bmatrix} U_1 + 2W_1 & 2\alpha(U_1 - 6V_1 + W_1) & 2(-U_1 + W_1) \\ 0 & V_1 & 0 \\ -U_1 + W_1 & \alpha(-2U_1 - 3V_1 + 5W_1) & 2U_1 + W_1 \end{bmatrix}.$$

Here

$$U_1 = -\frac{1}{\pi} \int_0^1 \zeta_0(u)\, du = 0.710446, \qquad V_1 = -\frac{1}{\pi} \int_0^1 \zeta(u)\, du = 0.997747, \qquad W_1 = -\frac{1}{\pi} \int_0^1 \zeta_1(u)\, du.$$

We recall that W_1 depends on the parameter γ, i.e., the Prandtl number. For a Prandtl number Pr $= 2/3$, we have $W_1 = 0.812276$. As $\gamma \to 0$, we have $W_1 \to U_1$, because $\zeta_1(u) \to \zeta_0(u)$.

4. Expansion in Eigenvectors. We seek a solution of problem (1.4), (1.5) as an expansion in the eigenvectors of the characteristic equation

$$h(x, \mu) = h_{as}(x, \mu) + A_0 h_0(x, \mu) + \int_0^1 \exp\Big(-\frac{x}{\eta}\Big) F(\eta, \mu) A(\eta)\, d\eta. \tag{4.1}$$

Here A_0 is an unknown constant and $A(\eta)$ is an unknown vector function with elements $A_j(\eta)$ ($j = 1, 2, 3$); the quantities ε_t and ε_n appearing in $h_{as}(x, \mu)$ are also unknowns.

Using boundary conditions (1.5), we reduce expansion (4.1) to the following vector singular integral equation with the Cauchy kernel:

$$h_{as}(0, \mu) + A_0 h_0(0, \mu) + \frac{1}{2} \int_0^1 \frac{\eta D(\mu, \eta)}{\eta - \mu} A(\eta)\, d\eta + \Lambda(\eta) A(\eta) = \mathbf{0}, \quad 0 < \mu < 1.$$

Let us introduce the auxiliary vector function

$$N(z) = \frac{1}{2} \int_0^1 \eta D(z, \eta) A(\eta) \frac{d\eta}{\eta - z} \tag{4.2}$$

and reduce the singular equation to the inhomogeneous vector boundary-value problem

$$P^+(\mu)[N^+(\mu) + h_{as}(0, \mu) + A_0 h_0(0, \mu)] = P^-(\mu)[N^-(\mu) + h_{as}(0, \mu) + A_0 h_0(0, \mu)], \qquad 0 < \mu < 1.$$

By means of the corresponding homogeneous problem (3.1), the inhomogeneous problem is reduced to the problem of determining the analytical vector function on from its zero jump on the cut:

$$[X^+(\mu)]^{-1}[N^+(\mu) + h_{as}(0,\mu) + A_0 h_0(0,\mu)]$$

$$= [X^-(\mu)]^{-1}[N^-(\mu) + h_{as}(0,\mu) + A_0 h_0(0,\mu)], \qquad 0 < \mu < 1. \tag{4.3}$$

Taking into account the features of the matrices and vectors included in Eq. (4.3), we shall obtain its general solution

$$N(z) = -h_{as}(0,z) - A_0 h_0(0,z) + X(z)[C + (z-\eta_0)^{-1}B], \tag{4.4}$$

where C and B are unknown vectors with constant elements c_j and b_j $(j = 1,2,3)$.

The pole for solution (4.4) at the point η_0 is eliminated by the condition

$$X(\eta_0)B + (1/2)A_0\eta_0 \operatorname{col}\{4\alpha, -1, \alpha\} = \mathbf{0},$$

whence

$$B = -\frac{1}{2}A_0\eta_0 X^{-1}(\eta_0)\operatorname{col}\{4\alpha,-1,\alpha\} = -\frac{A_0\eta_0}{2V(\eta_0)}\operatorname{col}\{4\alpha,-1,\alpha\}.$$

The auxiliary vector function (4.2) and the general solution (4.4) have a pole at the point $z = \infty$. We isolate the main parts of the expansions of these functions in a neighborhood of the point $z = \infty$. Equation (4.2) is written as

$$N(z) = \frac{1}{2}\int_0^1 \frac{\eta D_0(\eta z)}{\eta - z}A(\eta)\,d\eta - \frac{1}{2}\gamma\Big(z^2 - \frac{1}{3}\Big)\int_0^1 \frac{\eta D_1(\eta)}{\eta - z}A(\eta)\,d\eta.$$

It is easy to verify that $D_1(\eta)A(\eta) = a(\eta)\operatorname{col}\{2,0,1\}$, where $a(\eta) = a_1(\eta) + 5\alpha d(\eta)a_2(\eta) + a_3(\eta)$. Therefore, function (4.2) has the expansion

$$N(z) = zN^{(1)} + N^{(0)} + o(1), \qquad z \to \infty. \tag{4.5}$$

Here

$$N^{(1)} = J^{(1)}\operatorname{col}\{2,0,2\}, \qquad N^{(0)} = J^{(2)}\operatorname{col}\{0,1,0\} - J_2^{(2)}\operatorname{col}\{0,1,0\},$$

where

$$J^{(j)} = \frac{1}{2}\gamma\int_0^1 \eta^j a(\eta)\,d\eta \quad (j = 1,\ 2); \qquad J_2^{(2)} = \frac{3c}{2}\int_0^1 \eta^2 a_2(\eta)\,d\eta.$$

We now expand the right side of (4.4):

$$N(z) = -h_{as}(0,z) - A_0 h_0(0,z) + zC + B + X^{(0)}C + o(1), \qquad z \to \infty. \tag{4.6}$$

Comparing expansions (4.5) and (4.6), we obtain the following two systems of equations:

$$c_1 = 2c_3 + \frac{5}{2}k_t, \qquad c_2 = 2U - \frac{2}{3\sqrt{\pi}}k_t, \qquad c_3 = J^{(1)} - k_t \tag{4.7}$$

and

$$2J^{(2)} = -\varepsilon_n - \frac{1}{2}\varepsilon_t + b_1 - \frac{1}{3}U_1(c_1 + 2\alpha c_2 - 2c_3) + 4\alpha c_2 V_1 - \frac{2}{3}W_1(c_1 + 5\alpha c_2 + c_3),$$

$$-J_2^{(2)} = -\frac{1}{2}A_0 + b_2 - c_2V_1, \qquad J^{(2)} = -\varepsilon_t + b_3 + \frac{1}{3}U_1(c_1 + 2\alpha c_2 - 2c_3) + \alpha c_2 V_1 - \frac{1}{3}W_1(c_1 + 5\alpha c_2 + c_3). \tag{4.8}$$

From systems (4.7) and (4.8), we find all unknown coefficients of solution (4.4) and expansion (4.1). The unknown vector function $A(\eta)$ is found from Sokhotsky's formula applied to the vector function (4.2) into which solution (4.4) is substituted:

$$i\pi\eta D(\eta,\eta)A(\eta) = [X^+(\eta) - X^-(\eta)][C + (\eta - \eta_0)^{-1}B]. \tag{4.9}$$

Thus, all unknowns from expansion (4.1) are found.

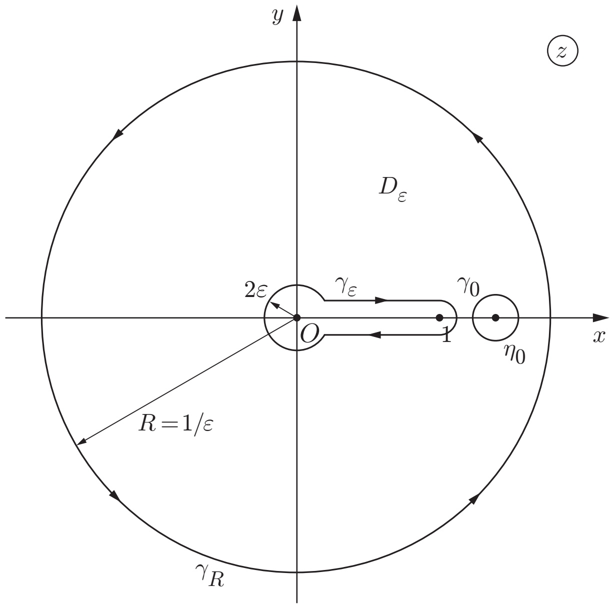

Fig. 1

5. Temperature and Concentration Jumps. Let us find all parameters of solution (4.4) and expansion (4.1) in explicit form. Because

$$c_1 + c_3 = 3J^{(1)} - k_t/2, \qquad c_1 - 2c_3 = 5k_t/2,$$

$$b_1 + 5\alpha b_2 + b_3 = 0, \qquad b_1 + 2\alpha b_2 - 2b_3 = 0,$$

equality (4.9) is representable as three scalar expansions:

$$i\pi\eta[a_1(\eta) + 4\alpha a_2(\eta) - 2p(\eta)a(\eta)] = \Big(\frac{5}{6}\,k_t + \frac{2}{3}\,\alpha c_2\Big)[U^+(\eta) - U^-(\eta)]$$

$$+\, 2\Big(J^{(1)} - \frac{1}{6}\,k_t + \frac{5}{3}\,\alpha c_2\Big)[W^+(\eta) - W^-(\eta)] - 4\alpha\Big(c_2 + \frac{b_2}{\eta - \eta_0}\Big); \tag{5.1}$$

$$3i\pi\eta c\eta^3 a_2(\eta) = \Big(c_2 + \frac{b_2}{\eta - \eta_0}\Big)[V^+(\eta) - V^-(\eta)], \qquad p(\eta) = \gamma\Big(\eta^2 - \frac{1}{3}\Big); \tag{5.2}$$

$$i\pi\eta[\alpha a_2(\eta) + a_3(\eta) - p(\eta)a(\eta)] = -\Big(\frac{5}{6}\,k_t + \frac{2}{3}\,\alpha c_2\Big)[U^+(\eta) - U^-(\eta)]$$

$$+\, \Big(J^{(1)} - \frac{1}{6}\,k_t + \frac{5}{3}\,\alpha c_2\Big)[W^+(\eta) - W^-(\eta)] - \alpha\Big(c_2 + \frac{b_2}{\eta - \eta_0}\Big)[V^+(\eta) - V^-(\eta)]. \tag{5.3}$$

According to (5.2), we obtain

$$J_2^{(2)} = \frac{1}{2\pi i}\int\limits_0^1 [V^+(\eta) - V^-(\eta)]\Big(c_2 + \frac{b_2}{\eta - \eta_0}\Big)\frac{d\eta}{\eta}.$$

To calculate this integral, we form the function

$$F(z) = [V(z) - z + V_1]\Big(\frac{c_2}{z} + \frac{b_2}{z(z - \eta_0)}\Big).$$

Let us consider a triply connected domain D_ε (Fig. 1) bounded by a complex contour consisting of a circumference γ_R of fairly large radius $R = 1/\varepsilon$ ($\varepsilon > 0$), a circumference γ_0 of radius ε with center at the point η_0, and a clockwise

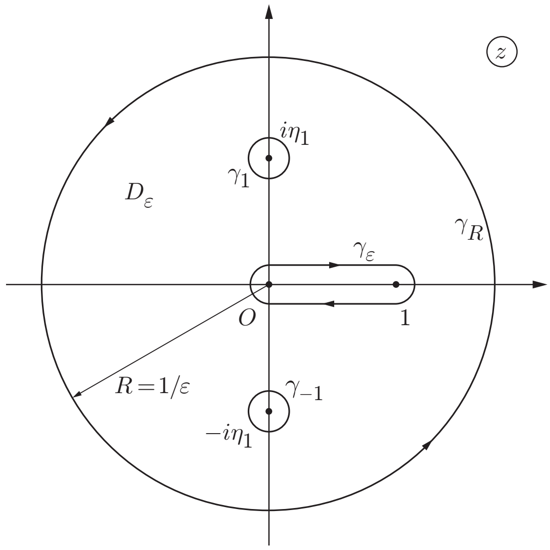

Fig. 2

contour γ_ε which separates from the cut $[0,1]$ by a distance ε and becomes a circumference of radius 2ε with center at the coordinate origin. According to the Cauchy theorem for multiconnected domains, we have

$$\frac{1}{2\pi i}\int\limits_{\gamma_R} F(z)\,dz = \frac{1}{2\pi i}\int\limits_{\gamma_0} F(z)\,dz - \frac{1}{2\pi i}\int\limits_{\gamma_\varepsilon} F(z)\,dz.$$

In this equality, we pass to the limit as $\varepsilon \to 0$. By virtue of the asymptotic relation $V(z) = z - V_1 + o(1)$ for $z \to \infty$, the integral over the circumference γ_R vanishes. As a result, we arrive at the equality

$$\frac{1}{2\pi i}\int\limits_0^1 [F^+(\eta) - F^-(\eta)]\,d\eta = \frac{1}{2\pi i}\int\limits_0^1 [V^+(\eta) - V^-(\eta)]\Big(\frac{c_2}{\eta} + \frac{b_2}{\eta(\eta-\eta_0)}\Big)\,d\eta \equiv J_2^{(2)} = \operatorname*{Res}_{\eta_0} F(z) + \operatorname*{Res}_{0} F(z).$$

Evaluation of these residues yields

$$J_2^{(2)} = V(0)(c_2 - b_2/\eta_0) + V_1 c_2 + V(\eta_0) b_2/\eta_0 - b_2.$$

Comparing this equality with the second in (4.8), we have $b_2 = \eta_0 c_2$ and $A_0 = 2V(\eta_0)c_2$. Hence, the vector B is finally constructed:

$$B = -\eta_0 c_2 \operatorname{col}\{4\alpha, -1, \alpha\}.$$

Combining Eq. (5.1) with Eq. (5.3) and with Eq. (5.2) multiplied by 5α, we have

$$i\pi\eta\gamma s(\eta)a(\eta) = (3J^{(1)} - k_t/2 + 5\alpha c_2)[W^+(\eta) - W^-(\eta)].$$

Consequently,

$$J^{(j)} = \Big(3J^{(1)} - \frac{1}{2}k_t + 5\alpha c_2\Big)J_j, \quad J_j = \frac{1}{2\pi i}\int\limits_0^1 [W^+(u) - W^-(u)]\,\frac{u^{j-1}\,du}{s(u)}, \quad j = 1, 2. \tag{5.4}$$

To calculate the integrals J_j, we form the functions

$$F_j(z) = \frac{W(z) - z + W_1}{s(z)}\,z^{j-1} \qquad (j = 1, 2),$$

which are analytic in a four-connected domain D_ε (Fig. 2) bounded by a complex contour. This contour consists of a circumference γ_R of fairly large radius $R = 1/\varepsilon$ ($\varepsilon > 0$), two circumferences γ_1: $|z - i\eta_1| = \varepsilon$ and γ_{-1}: $|z + i\eta_1| = \varepsilon$, and a contour γ_ε that encloses the cut $[0,1]$ clockwise and is separated from it by a distance ε. According to the Cauchy theorem for multiconnected domains,

$$\frac{1}{2\pi i}\int\limits_{\gamma_R} F_j(z)\,dz = \operatorname*{Res}_{i\eta_1} F_j(z) + \operatorname*{Res}_{-i\eta_1} F_j(z) - \frac{1}{2\pi i}\int\limits_{\gamma_\varepsilon} F_j(z)\,dz.$$

In this equality, we pass to the limit as $\varepsilon \to 0$. By virtue of the asymptotic form $F_j(z) = o(z^{j-3})$ ($j = 1,\ 2$), we find that the integral on the left side of the previous equality vanishes. We have

$$\frac{1}{2\pi i}\int\limits_0^1 [F_j^+(\eta) - F_j^-(\eta)]\,d\eta \equiv J_j = \operatorname*{Res}_{i\eta_1} F_j(z) + \operatorname*{Res}_{-i\eta_1} F_j(z).$$

Consequently,

$$J_1 = \operatorname*{Res}_{i\eta_1} \frac{W(z) - z + W_1}{s(z)} + \operatorname*{Res}_{-i\eta_1} \frac{W(z) - z + W_1}{s(z)} = -\frac{1}{6i\eta_1}\,[W(i\eta_1) - W(-i\eta_1) - 2i\eta_1],$$

$$J_2 = \operatorname*{Res}_{i\eta_1} \frac{W(z) - z + W_1}{s(z)} z + \operatorname*{Res}_{-i\eta_1} \frac{W(z) - z + W_1}{s(z)}\, z = -\frac{1}{6}\,[W(i\eta_1) + W(-i\eta_1) + 2W_1].$$

Now from Eqs. (5.4), we obtain

$$J^{(1)} = \frac{J_1}{1 - 3J_1}\Big(5\alpha c_2 - \frac{k_t}{2}\Big), \qquad J^{(2)} = \frac{J_2}{1 - 3J_1}\Big(5\alpha c_2 - \frac{k_t}{2}\Big).$$

Thus, we found all parameters of solution (4.4). From the first and third equations (4.8), we derive the formulas

$$\begin{aligned} \varepsilon_t &= \alpha c_2\Big[V_1 - \eta_0 + \frac{2}{3}\,U_1 - \frac{5}{3}\,W_1 - 5\,\frac{J_2 + J_1 W_1}{1 - 3J_1}\Big] + k_t\Big[\frac{5}{6}\,U_1 + \frac{W_1 + 3J_2}{6(1 - 3J_1)}\Big], \\ \varepsilon_n &= 3\varepsilon_t/2 + 2\alpha c_2(V_1 - \eta_0 - U_1) - 5k_t U_1/2. \end{aligned} \tag{5.5}$$

The boundary-value problem (1.4), (1.5) is completely solved.

6. Numerical Calculations and Discussion of Results. Formulas (5.5) can be written in standard form $\varepsilon_t = T_t k_t + T_u(2U)$ and $\varepsilon_n = N_t k_t + N_u(2U)$. The temperature and concentration jump coefficients are evaluated from the formulas

$$\begin{aligned} T_t &= (3\eta_0 - 3V_1 + 18U_1 + 10W_1 - W_1')/24, \\ T_u &= \alpha[-3\eta_0 + 3V_1 + 2U_1 - 10W_1 + 5W_1']/3, \\ N_t &= [7\eta_0 - 7V_1 - 18U_1 + 10W_1 - W_1']/16, \\ N_u &= \alpha[-7\eta_0 + 7V_1 - 2U_1 - 10W_1 + W_1']/2. \end{aligned} \tag{6.1}$$

Here

$$W_1' = -i\eta_1\,\frac{W(i\eta_1) + W(-i\eta_1)}{W(i\eta_1) - W(-i\eta_1)}$$

and $W_1' \to W_1$ as $\gamma \to 0$. We note that when $\gamma \to 0$ (the ES equation becomes the BGK equation), formula (5.5) becomes the corresponding formulas derived from the BGK equation:

$$\varepsilon_t = k_t(\eta_0 - V_1 + 9U_1)/8 + (2U)\alpha(-\eta_0 + V_1 - U_1),$$

$$\varepsilon_n = k_t(7\eta_0 - 7V_1 - 9U_1)/16 + (2U)7\alpha(-\eta_0 + V_1 - U_1)/2.$$

We note that the Prandtl number differs somewhat from the value of 2/3. For rigid sphere molecules, $\mathrm{Pr} = 0.66072$ [14] and, hence, $\gamma_0 = -0.483427$. Numerical calculations using the above formulas for the value of γ_0 corresponding to the given Prandtl number lead to the following results:

$$T_t(\gamma_0) = 0.826285, \qquad T_u(\gamma_0) = -1.09308,$$

$$N_t(\gamma_0) = -0.35851, \qquad N_u(\gamma_0) = -0.93760.$$

In [9], for a BGK model with a collision frequency proportional to the molecular velocity (i.e., with a constant free path length of molecules), the following results were obtained: $T_t = 0.79954$, $T_u = -1.0239$, $N_t = -0.39863$, and $N_u = -0.82905$.

Let us convert to dimensional quantities. We note that in temperature jump problems, it is customary to use the definition of the free path length for molecules in terms of thermal conductivity (thermal diffusivity) [15]. We use the definition of the free path length that coincides with the corresponding definition according to [4] for Pr $= 2/3$:

$$l = \frac{2\chi}{3}\sqrt{\frac{2kT}{\pi m}},$$

where χ is the thermal diffusivity.

Then, the expression for the temperature jump is written as

$$\varepsilon_= C_t l \left(\frac{dT}{dx}\right)_\infty .$$

In this case, the temperature jump coefficient obtained in the present study has a value $C_t = 2.06571$. We recall that the ES equation with a constant collision frequency [7] yields $C_t = 2.20576$.

For comparison, numerical calculations using the complete Boltzmann equation for rigid sphere molecules [15] yielded $C_t = 2.1113$, and in [16], $C_t = 2.20711$ was obtained for a 13-moment kinetic model with a constant frequency of molecular collisions. A numerical study [3] of the model of rigid sphere molecules with a variable collision frequency using the discrete coordinate method yielded the value $C_t = 2.0421$. We note that the results given in the cited papers are converted taking into account the definition of the free path length of gas molecules adopted in the present paper.

Conclusions. We shall dwell on the distinctive features of the analytical solution described above. Decomposition of the distribution function reduces the Smoluchowski problem to a typical vector transfer equation with a 3×3 matrix kernel. There are no exact solutions of equations with such kernels. An exception is a study [9], in which the same problem for the BGK equation was considered. One of the central points that ensure an analytical solution is diagonalization of the Riemann–Hilbert matrix vector boundary-value problem, to which the initial boundary-value reduces. The matrix coefficient of the Riemann–Hilbert problem has singularities — simple poles on the imaginary axis. When the ES equation becomes the BGK equation ($\gamma \to 0$), these poles disappear — move to infinity on the imaginary axis. For the first time, for analytical methods under resolvability conditions for the general solution of the Riemann–Hilbert problem, the values of the factor matrix were used not only at points of the discrete spectrum but also at the above-mentioned poles.

The results obtained in the present paper can be useful to analyze the behavior of aerosol particles in nonuniformly heated gases and to solve various problems of the kinetic theory of gas and plasma, the theory of neutron and electron transport, theoretical astrophysics, and other areas.

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